Deriving Quantum EM Solutions Using Classical Knowledge
--An Overview

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University of Bologna, Italy, 1088 till now
Nicolaus Copernicus, 1473-1543
Gugliemo Marconi, 1874-1937

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Why Quantum Electromagnetics/Optics?

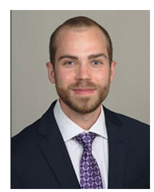
Importance of Quantum Electromagnetics
When wavelength much longer than atomic spacings—E.g. microwave photons (macroscopic quantum electromagnetics)

- Leaps and bounds progress of quantum technologies:
  - Quantum communications—security and encryption
  - Quantum computers—quantum parallelism
  - Quantum sensing—enhanced sensitivity
- These are going to be transformative technologies

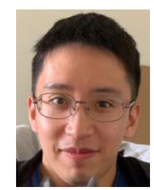

Aiyin Liu


Wei Sha


Carlos Salazar-Lazaro


Thomas Roth


Dong-Yeop Na


Chris J. Ryu


Erhan Kudeki

Classical EM knowledge has been with us for over 150 years.
Wish for a gradual transition from classical to quantum!

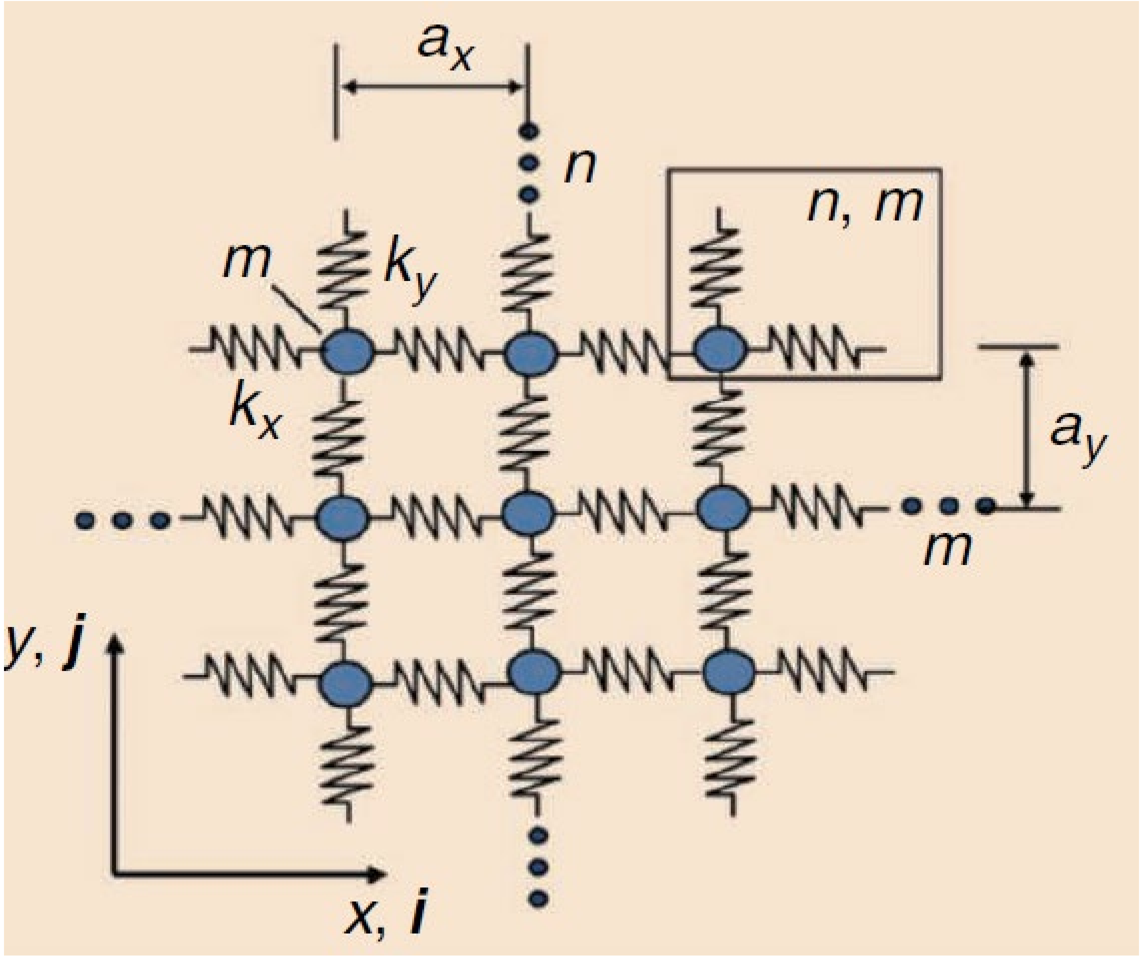
Short courses in AP-S, URSI Conferences

Lectures 38, 39

Lectures on
Electromagnetic Field Theory:
A Comprehensive Overview

Weng Cho Chew

Spring 2025,[1] Purdue University



Electromagnetics waves is analogous to elastic waves: masses connected by springs.

Classical CEM and Quantum CEM

Both Classical and Quantum Maxwell's equations are derivable from energy conservation!

Classical Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$H = \int \left(\frac{\mathbf{D}^2}{\varepsilon} + \frac{\mathbf{B}^2}{\mu} \right) dV$$

Quantum Maxwell's equations

$$\nabla \times \hat{\mathbf{E}} = -\frac{\partial \hat{\mathbf{B}}}{\partial t}$$

$$\nabla \times \hat{\mathbf{H}} = \frac{\partial \hat{\mathbf{D}}}{\partial t} + \hat{\mathbf{J}}$$

$$\nabla \cdot \hat{\mathbf{D}} = \hat{\rho}$$

$$\nabla \cdot \hat{\mathbf{B}} = 0$$

Fields become quantum operators. $\hat{H} = \int \left(\frac{\hat{\mathbf{D}}^2}{\varepsilon} + \frac{\hat{\mathbf{B}}^2}{\mu} \right) dV$

An operator must act on a state vector (quantum state function). $\hat{H}|\psi\rangle = i\hbar\partial_t|\psi\rangle$

Cohen-Tannoudji et al (1986), Mendel and Wolf (1995).

Quantization of EM field: Glauber (1991), Huttner-Barnett (1992), Milonni (1995), Welsch (1996), Suttorp and Wub (2004), Scheel and Buhmann (2008), Philbin (2010), A. Drezet (2017), Jauslin {2019).

Chew et al (2016).

Chew et al (2019).

Chew et al (Feb 2021, IEEE AP Magazine).

Classical Theory and Hamiltonian Theory

- Illustrate with a simple pendlum: Simple Hamiltonian mechanics

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$$

$$T = \frac{p^2}{2m}$$ Kinetic Energy

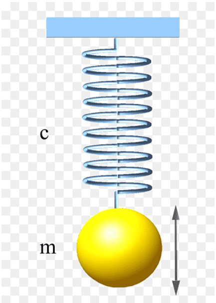
$$V = \frac{1}{2}\kappa q^2 = \frac{1}{2} m\omega_0^2 q^2$$ Potential Energy

- *p*, *q* are dynamic conjugate variables:

$$\frac{d}{dt} H\left(p(t), q(t)\right) = 0 = \frac{dp}{dt}\frac{\partial H}{\partial p} + \frac{dq}{dt}\frac{\partial H}{\partial q}$$

$$\frac{dp}{dt} = -C\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = C\frac{\partial H}{\partial p}.$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$ Hamilton equations



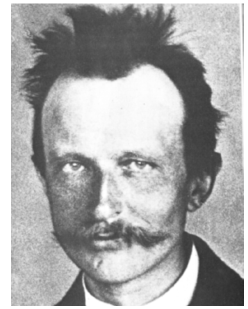
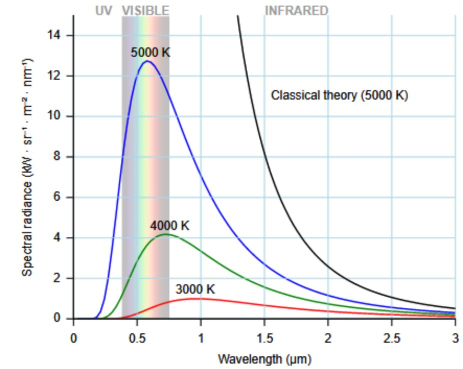
Equations of motion: same as from Newton’s law

$$m\frac{d^2 q}{dt^2} = -m\omega_0^2 q = -\kappa q$$

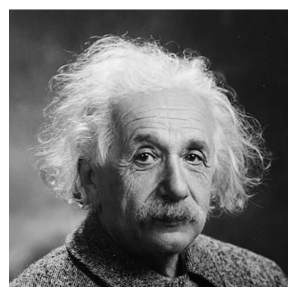
The Need for a New Theory that is Quantum

- Planck's radiation law (1900).

$$E = \hbar\omega$$

- Photoelectric effect (Hertz, 1887, Einstein, 1905).

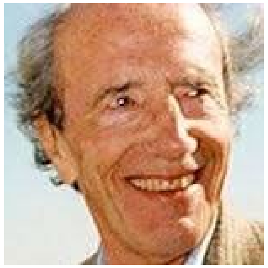



- Wave-particle duality, De Broglie, 1923.

$$p = \hbar k$$

The Birth of Schrodinger Equation

Story according to
Francis E Low of MIT



$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2 = E$$

Classical Hamiltonian

$$p = \hbar k \qquad E = \hbar\omega$$

De Broglie hypothesis and Planck's law

$$\frac{\hbar^2 k^2}{2m} + \frac{1}{2} m\omega_0^2 q^2 = \hbar\omega$$

$$ik = \frac{\partial}{\partial q} \qquad -i\omega = \frac{\partial}{\partial t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\Psi(q,t) + \frac{1}{2} m\omega_0^2 q^2 \Psi(q,t) = i\hbar\frac{\partial}{\partial t}\Psi(q,t)$$

$$\Psi(q,t) \sim e^{ikq - i\omega t}$$

Schrodinger Equation in Operator Form

KE PE Total E

$$\boxed{\frac{\hbar^2 k^2}{2m} + \frac{1}{2} m\omega_0^2 q^2} = \hbar\omega$$ ← Classical Hamiltonian

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b}$$

$$\overline{\mathbf{u}}^\dagger \cdot \overline{\mathbf{u}} = \overline{\mathbf{I}}, \quad \text{Resolution of Identity Operator}$$

$$\overline{\mathbf{u}} \cdot \overline{\mathbf{A}} \cdot \overline{\mathbf{u}}^\dagger \cdot \overline{\mathbf{u}} \cdot \mathbf{x} = \overline{\mathbf{u}} \cdot \mathbf{b} \Longrightarrow \overline{\mathbf{A}}' \cdot \mathbf{x}' = \mathbf{b}'$$

$$\hbar k \Rightarrow -i\hbar\frac{\partial}{\partial q} \qquad -i\omega \Rightarrow \frac{\partial}{\partial t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\Psi(q,t) + \frac{1}{2}m\omega_0^2 q^2 \Psi(q,t) = i\hbar\frac{\partial}{\partial t}\Psi(q,t)$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial q} \qquad \hat{q} = \hat{I}q \qquad [\hat{p},\hat{q}] = \hat{p}\hat{q} - \hat{q}\hat{p} = -i\hbar\hat{I}$$

$$\frac{\hat{p}^2}{2m}\Psi(q,t) + \frac{1}{2}m\omega_0^2\hat{q}^2\Psi(q,t) = i\hbar\frac{\partial}{\partial t}\Psi(q,t)$$

State Vector

$$\hat{H}\Psi(q,t) = i\hbar\frac{\partial}{\partial t}\Psi(q,t) \qquad \hat{H}|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle$$

State Variable Approach

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{q}^2$$ ← Hamiltonian Operator

Separation of Variables (A Sturm-Liouville Problem)

$$\Psi(q,t) = \Psi_n(q) e^{-i\omega_n t}$$

Eigenfunction

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{1}{2}m\omega_0^2 q^2\right]\Psi_n(q) = E_n \Psi_n(q)$$

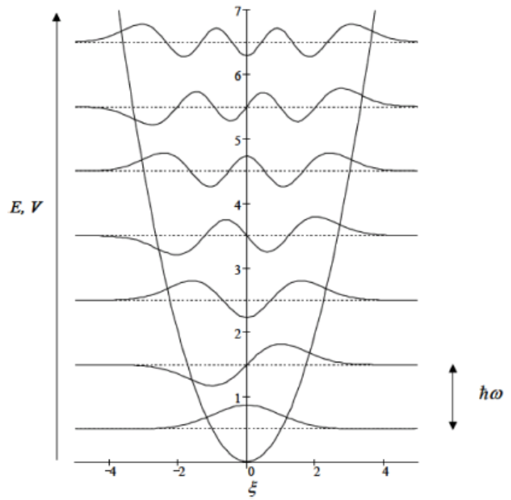
Eigenvalue

$$E_n = \hbar\omega_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$$

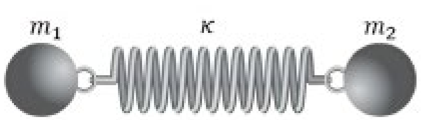
Photon Number State

$$\Psi_n(q) = \sqrt{\frac{1}{2^n n!}\sqrt{\frac{m\omega_0}{\pi\hbar}}} e^{-\frac{m\omega_0}{2\hbar}q^2} H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}q\right)$$

Fock State



Analog of Fourier Harmonics



Needed to describe the physics of a quantum pendulum

Probabilistic Interpretation of Quantum Theory

- Schrodinger postulated his equation motivated by the works of his predecessors, Newton, Planck, and De Broglie.
- But he didn't know the physical meaning of Schrodinger wave function in his equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\Psi(q,t)+\frac{1}{2}m\omega_0^2 q^2\Psi(q,t)=i\hbar\frac{\partial}{\partial t}\Psi(q,t)$$

- It was Max Born who associated the wave function with probability distribution function (PDF).

Probabilistic Interpretation of Wave Functions

$$\int_{-\infty}^{\infty} dq|\Psi(q,t)|^2 = 1 \Leftrightarrow \underbrace{\langle\Psi(t)|\Psi(t)\rangle}_{\text{Dirac}} = 1 \Leftrightarrow \underbrace{\mathbf{\Psi}(t)^\dagger \cdot \mathbf{\Psi}(t)}_{\text{Matrix}} = 1$$

$$\int_{-\infty}^{\infty} dqq|\Psi(q,t)|^2 = \langle q(t)\rangle = \bar{q}(t) \Leftrightarrow \underbrace{\langle\Psi(t)|\hat{q}|\Psi(t)\rangle}_{\text{Dirac}} \Leftrightarrow \underbrace{\mathbf{\Psi}(t)^\dagger \cdot \overline{\mathbf{q}} \cdot \mathbf{\Psi}(t)}_{\text{Matrix}}$$

$$\hat{q} = \hat{I}q$$

$$-i\hbar\int_{-\infty}^{\infty} dq\Psi^*(q,t)\frac{\partial}{\partial q}\Psi(q,t) = \langle p(t)\rangle = \bar{p}(t)$$

$$\Leftrightarrow \underbrace{\langle\Psi(t)|\hat{p}|\Psi(t)\rangle}_{\text{Dirac}} \Leftrightarrow \underbrace{\mathbf{\Psi}(t)^\dagger \cdot \overline{\mathbf{p}} \cdot \mathbf{\Psi}(t)}_{\text{Matrix}}$$

$$\hat{p} = -i\hbar\partial/(\partial q).$$

Leslie Ballentine, Q. Mechanics, A Modern Dev. p. 50: What is a Quantum Observable?

…Dirac, to whom we are indebted for so much of modern formulation of QM, used the word indiscriminately…

Chinese Adage: If you believe in your book completely, it is better not to have books; if you believe in your teacher completely, it is better not to have teachers.

A dynamic variable in the quantum world becomes a random variable. It is the hallmark of a **quantum observable.**

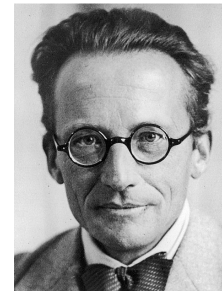
It is characterized by an operator-vector pair.

$$p \Leftrightarrow \{\hat{p}, \Psi(q,t)\}\,, \qquad q \Leftrightarrow \{\hat{q}, \Psi(q,t)\}$$

$$p \Leftrightarrow \{\hat{p}, |\Psi(t)\rangle\}\,, \qquad q \Leftrightarrow \{\hat{q}, |\Psi(t)\rangle\}$$

Heisenberg Picture vs Schrodinger Picture

Solvable in closed form $\hat{H}|\Psi\rangle = i\hbar\partial_t|\Psi\rangle$



$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$$

$$\bar{O}(t) = \langle\Psi(t)|\hat{O}_s|\Psi(t)\rangle$$ ⇦ Schrodinger Picture

$$\bar{O}(t) = \langle\Psi(0)|e^{i\hat{H}t/\hbar}\hat{O}_s e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle$$ ⇦ Heisenberg Picture

In the above, $\bar{O}(t)$ is unique but $\langle\Psi(t)|, \hat{O}_s, |\Psi(t)\rangle$ are not unique.

$$\bar{O}(t) = \langle\Psi(0)|\hat{O}_h(t)|\Psi(0)\rangle \qquad \hat{O}_h(t) = e^{i\hat{H}t/\hbar}\hat{O}_s e^{-i\hat{H}t/\hbar}$$

Homomorphism—Heisenberg Equations of Motion

$$\hat{p}(t) = e^{-\frac{i}{\hbar}\hat{H}t}\,\hat{p}\,e^{\frac{i}{\hbar}\hat{H}t}$$

Repeated use of fundamental commutator:

$$[\hat{q},\hat{p}] = i\hbar\hat{I},$$

$$\frac{d\hat{p}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{p},\hat{H}\right], \qquad \frac{d\hat{q}(t)}{dt} = \frac{i}{\hbar}\left[\hat{q},\hat{H}\right]$$

$$[\hat{p},\hat{q}^n] = -in\hat{q}^{n-1}\hbar = -i\hbar\left(\frac{\partial}{\partial\hat{q}}\hat{q}^n\right)$$

$$\left[\hat{p},\hat{H}\right] = -i\hbar\frac{\partial}{\partial\hat{q}}H(\hat{p},\hat{q})$$

$$\frac{d\hat{p}}{dt} = -\frac{\partial\hat{H}(\hat{p},\hat{q})}{\partial\hat{q}}, \qquad \frac{d\hat{q}}{dt} = \frac{\partial\hat{H}(\hat{p},\hat{q})}{\partial\hat{p}}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

Quantum Hamilton equations. Classical Hamilton equations.

- Homomorphism seen between classical and quantum Hamilton equations.
- Commutator induces partial derivatives.

Sir William Rowan Hamilton (1805–1865)

W.H. Louisell, "Quantum statistical properties of radiation." (1973).
K. Gottfried, Kurt, and TM. Yan, *Quantum mechanics: fundamentals*. Vol. 2. New York: Springer, 2004.
W. C. Chew, A.Y. Liu, C. Salazar-Lazaro, W.E.I. Sha, "Quantum electromagnetics: A new look—Parts I & II." *IEEE Journal on Multiscale and Multiphysics Computational Techniques* 1 (2016): 85-97.

Hamiltonian—Classical Picture

$$H = \frac{1}{2}\int\limits_V d\mathbf{r}\left[\varepsilon \mathbf{E}^2(\mathbf{r},t) + \frac{\mathbf{B}^2(\mathbf{r},t)}{\mu}\right]$$

$$\nabla \cdot \mathbf{A} = 0 \text{ with } \Phi = 0$$

$$H = \frac{1}{2}\int\limits_V d\mathbf{r}\left[\varepsilon \dot{\mathbf{A}}^2(\mathbf{r},t) + \frac{\left(\nabla \times \mathbf{A}(\mathbf{r},t)\right)^2}{\mu}\right].$$

$$H = \frac{1}{2}\int\limits_V d\mathbf{r}\left[\frac{\mathbf{\Pi}^2(\mathbf{r},t)}{\varepsilon} + \frac{\left(\nabla \times \mathbf{A}(\mathbf{r},t)\right)^2}{\mu}\right]$$

Derivation of Classical ME from Hamiltonian Theory

Total Hamiltonian $$H = \frac{1}{2} \int_V d\mathbf{r} \left[\frac{\mathbf{\Pi}^2(\mathbf{r},t)}{\varepsilon} + \frac{(\nabla \times \mathbf{A}(\mathbf{r},t))^2}{\mu} \right] = \int_V \mathcal{H} d\mathbf{r}$$

$\mathcal{H}$ ← Hamiltonian density

$$\frac{\delta H}{\delta \mathbf{\Pi}} = \frac{\partial \mathcal{H}}{\partial \mathbf{\Pi}} \qquad \frac{\delta H}{\delta \mathbf{A}} = \frac{\partial \mathcal{H}}{\partial \mathbf{A}}$$

Functional derivatives
And partial derivatives

$$\frac{\delta H}{\delta \mathbf{A}} = \frac{\partial \mathcal{H}}{\partial \mathbf{A}} = \frac{\partial H}{\partial \mathbf{\Pi}} = \frac{\mathbf{\Pi}}{\varepsilon} = \dot{\mathbf{A}}$$

The conjugate dynamic variables, $\mathbf{\Pi}$ and $\mathbf{A}$ vary in tandem to keep the total Hamiltonian H a constant.

$$\frac{\delta H}{\delta \mathbf{A}} = \frac{\partial \mathcal{H}}{\partial \mathbf{A}} = \frac{\partial H}{\partial \mathbf{A}} = \frac{1}{\mu} \nabla \times \nabla \times \mathbf{A} = -\frac{\partial \mathbf{\Pi}}{\partial t} = -\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \varepsilon \partial_t \mathbf{E}(\mathbf{r},t)$$
$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\mu \partial_t \mathbf{H}(\mathbf{r},t)$$

Classical Maxwell’s Equations!

Rehash of the Previous Slide

$$\hat{p}(t) = e^{-\frac{i}{\hbar}\hat{H}t}\,\hat{p}\,e^{\frac{i}{\hbar}\hat{H}t}$$

Repeated use of fundamental commutator:

$$[\hat{q}, \hat{p}] = i\hbar\hat{I},$$

$$\frac{d\hat{p}(t)}{dt} = -\frac{i}{\hbar}\left[\hat{p}, \hat{H}\right], \qquad \frac{d\hat{q}(t)}{dt} = \frac{i}{\hbar}\left[\hat{q}, \hat{H}\right]$$

$$\left[\hat{p}, \hat{H}\right] = -i\hbar\frac{\partial}{\partial\hat{q}}H(\hat{p}, \hat{q})$$

$$\frac{d\hat{p}}{dt} = -\frac{\partial\hat{H}(\hat{p}, \hat{q})}{\partial\hat{q}}, \qquad \frac{d\hat{q}}{dt} = \frac{\partial\hat{H}(\hat{p}, \hat{q})}{\partial\hat{p}}$$

$\Longleftrightarrow$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

Hamiltonian—Quantum Picture

$$\hat{H} = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon\hat{\mathbf{E}}^2(\mathbf{r},t) + \frac{\hat{\mathbf{B}}^2(\mathbf{r},t)}{\mu}\right] = \frac{1}{2}\int_V d\mathbf{r}\left[\frac{\hat{\mathbf{\Pi}}^2(\mathbf{r},t)}{\varepsilon} + \frac{1}{\mu}\left(\nabla\times\hat{\mathbf{A}}(\mathbf{r},t)\right)^2\right]$$

$$[\hat{q}_i(t), \hat{p}_j(t)] = i\hbar\delta_{ij}\hat{I}$$

$$\left[\hat{p}_{i'}, \hat{H}\right] = -i\hbar\frac{\delta\hat{H}}{\delta\hat{q}_{i'}} = i\hbar\partial_t\hat{p}_{i'}(t)$$

$$\left[\hat{q}_{i'}, \hat{H}\right] = i\hbar\frac{\delta\hat{H}}{\delta\hat{p}_{i'}} = i\hbar\partial_t\hat{q}_{i'}(t)$$

$$\left[\hat{\mathbf{\Pi}}(\mathbf{r}',t), \hat{H}\right] = -i\hbar\frac{\delta\hat{H}}{\delta\hat{\mathbf{A}}(\mathbf{r}',t)} = i\hbar\partial_t\hat{\mathbf{\Pi}}(\mathbf{r}',t)$$

$$\left[\hat{\mathbf{A}}(\mathbf{r}',t), \hat{H}\right] = i\hbar\frac{\delta\hat{H}}{\delta\hat{\mathbf{\Pi}}(\mathbf{r}',t)} = i\hbar\partial_t\hat{\mathbf{A}}(\mathbf{r}',t)$$

$$\nabla\times\hat{\mathbf{H}}(\mathbf{r},t) = \varepsilon\partial_t\hat{\mathbf{E}}(\mathbf{r},t)$$

$$\nabla\times\hat{\mathbf{E}}(\mathbf{r},t) = -\mu\partial_t\hat{\mathbf{H}}(\mathbf{r},t)$$

Quantum Maxwell's Equations!

More on Quantum Maxwell's Equations…

$$\hat{H}|\Psi(t)\rangle = i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t}$$

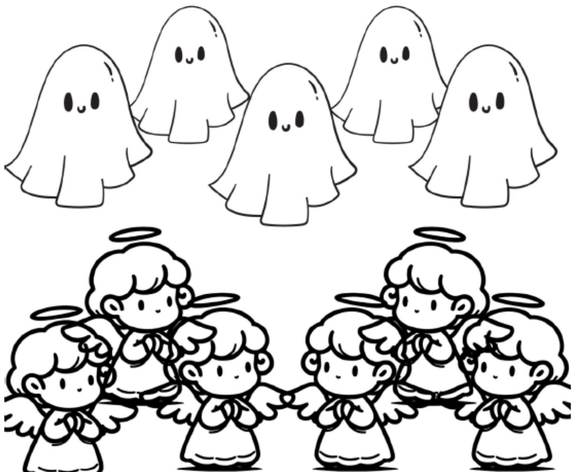
$$\nabla\times\hat{\mathbf{H}}(\mathbf{r},t) = \varepsilon\partial_t\hat{\mathbf{E}}(\mathbf{r},t)$$

$$\nabla\times\hat{\mathbf{E}}(\mathbf{r},t) = -\mu\partial_t\hat{\mathbf{H}}(\mathbf{r},t)$$

$$\nabla\times\hat{\mathbf{H}}(\mathbf{r},t)|\Psi(0)\rangle = \varepsilon\partial_t\hat{\mathbf{E}}(\mathbf{r},t)|\Psi(0)\rangle$$

$$\nabla\times\hat{\mathbf{E}}(\mathbf{r},t)|\Psi(0)\rangle = -\mu\partial_t\hat{\mathbf{H}}(\mathbf{r},t)|\Psi(0)\rangle$$

$$\hat{\mathbf{H}}(\mathbf{r},t)|\Psi(0)\rangle \qquad \hat{\mathbf{E}}(\mathbf{r},t)|\Psi(0)\rangle$$



Min Chen of MIT

Ghost-Angel interpretation

Quantum parallelism:

Quantum Fourier Transform: $(\log n)^2$

Operators in Normalized Coordinates

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) = \xi\hat{I} \quad \Longleftrightarrow \quad \hat{q} = \hat{I}q$$

$$\hat{\pi} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) = -i\frac{d}{d\xi} \quad \Longleftrightarrow \quad \hat{p} = -i\hbar\partial/(\partial q)$$

Ladder Operators

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-i\hat{\pi} + \hat{\xi}\right)$$

Raising Operator

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(i\hat{\pi} + \hat{\xi}\right)$$

Lowering Operator

Schrodinger Equation in Normalized Coordinates

$$\hat{H}\Psi_n(q) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{1}{2}m\omega_0^2 q^2\right]\Psi_n(q) = E_n\Psi_n(q).$$

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right)\Psi_n(\xi) = \frac{E_n}{\hbar\omega_0}\Psi_n(\xi)$$

$$\frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)\Psi_n(\xi) = \left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\Psi_n(\xi) = \frac{E_n}{\hbar\omega_0}\Psi_n(\xi)$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I}$$

Source of quantum interference

Coherent State (What is it?)

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\hat{a}|\Psi_n\rangle = \sqrt{n}|\Psi_{n-1}\rangle$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

This state remains coherent after being operated upon by the annihilation operator, or the lowering operator.

Time-Evolution of a Coherent State

$$|\Psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\Psi(0)\rangle \qquad |\alpha, t\rangle = e^{-i\omega_0 t/2}|\alpha e^{-i\omega_0 t}\rangle = e^{-i\omega_0 t/2}|\tilde{\alpha}\rangle$$

More on the Coherent State

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}^{\dagger} + \hat{a}\right)$$ "Re"=?

$$\hat{\pi} = \frac{i}{\sqrt{2}}\left(\hat{a}^{\dagger} - \hat{a}\right)$$ "Im"=?

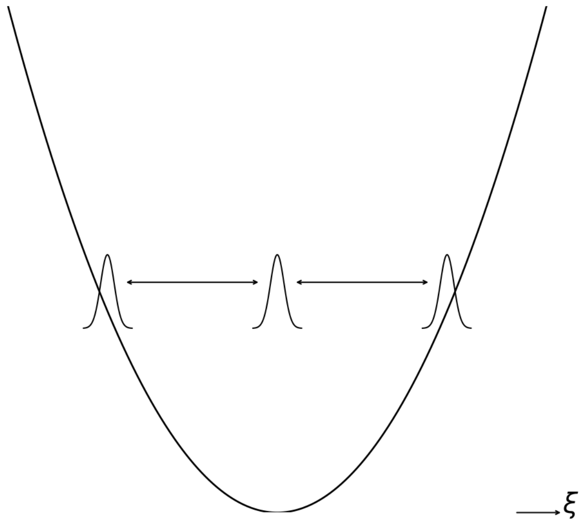
$$\bar{\xi} = \langle \xi \rangle = \langle \alpha | \hat{\xi} | \alpha \rangle = \frac{1}{\sqrt{2}}\left(\alpha^* + \alpha\right)\underbrace{\langle \alpha | \alpha \rangle}_{=1} = \sqrt{2}\Re e\left(\alpha\right) \neq 0$$

$$\bar{\pi} = \langle \pi \rangle = \langle \alpha | \hat{\pi} | \alpha \rangle = \frac{i}{\sqrt{2}}\left(\alpha^* - \alpha\right)\underbrace{\langle \alpha | \alpha \rangle}_{=1} = \sqrt{2}\Im m\left(\alpha\right) \neq 0$$

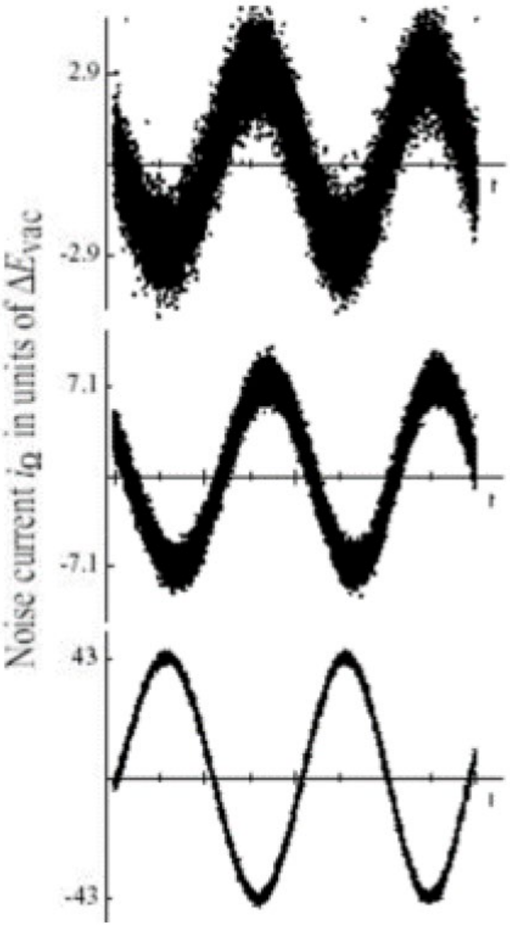
The Correspondence Principle

$$\bar{\xi}(t) = \langle \xi(t) \rangle = \sqrt{2}|\alpha| \cos(\omega_0 t + \Psi)$$

$$\bar{\pi}(t) = \langle \pi(t) \rangle = -\sqrt{2}|\alpha| \sin(\omega_0 t + \Psi)$$



Coherent State is a Quantum Observable



A quantum observable is a random variable.
It is characterized by an operator-vector pair.

$$\pi \equiv \{\hat{\pi}, |\alpha\rangle\}$$

$$\xi \equiv \left\{\hat{\xi}, |\alpha\rangle\right\}$$

Consider a Single-Mode Plane Wave—Semiclassical Case

$$H = \frac{1}{2}\int d\mathbf{r}\left[\varepsilon\mathbf{E}^2(\mathbf{r},t) + \frac{\mathbf{B}^2(\mathbf{r},t)}{\mu}\right.$$

$$A_x(z,t) = \sum_l \frac{1}{2}A_l e^{-i\omega_l t + ik_l z} + c.c.$$ Multi-mode case

$$A_x(z,t) = \frac{1}{2}A_l e^{-i\omega_l t + ik_l z} + c.c.$$ Single-mode case

$$H_l = n_l \hbar\omega_l$$

$$\left|\frac{A_l}{2}\right|^2 = \frac{n_l \hbar\omega_l}{V\varepsilon\omega_l^2}$$

Consider a Single-Mode Plane Wave—Quantum Case

$$\hat{H} = \frac{1}{2}\int_V d\mathbf{r}\left[\varepsilon\hat{\mathbf{E}}^2(\mathbf{r},t) + \frac{\hat{\mathbf{B}}^2(\mathbf{r},t)}{\mu}\right]$$

$$\hat{\mathbf{A}}(z,t) = \hat{x}\hat{A}_x(z,t) = \hat{x}\sum_l \frac{1}{2}\hat{A}_l e^{-i\omega_l t + ik_l z} + h.c.$$ Multi-mode case

$$\hat{\mathbf{A}}(z,t) = \hat{x}\hat{A}_x(z,t) = \hat{x}\frac{1}{2}\hat{A}_l e^{-i\omega_l t + ik_l z} + h.c.$$ Single-mode case

$$\hat{H}_l = \frac{1}{4}V\varepsilon\omega_l^2(\hat{A}_l\hat{A}_l^\dagger + \hat{A}_l^\dagger\hat{A}_l)$$

$$\frac{1}{2}\sqrt{V\varepsilon\omega_l^2}\hat{A}_l = \sqrt{\frac{\hbar\omega_l}{2}}\hat{a}_l$$

$$\hat{H}_l = \frac{1}{2}\hbar\omega_l\left(\hat{a}_l\hat{a}_l^\dagger + \hat{a}_l^\dagger\hat{a}_l\right) = \hbar\omega_l\left(\hat{a}_l^\dagger\hat{a}_l + \frac{1}{2}\right)$$

Quantum State Equation

$$\hat{H}_l|\Psi_l\rangle = i\hbar\partial_t|\Psi_l\rangle$$

$$\hat{A}_x(z,t) = \sqrt{\frac{\hbar}{2V\varepsilon\omega_l}}e^{i\theta_l}e^{ik_l z - i\omega_l t}\hat{a}_l + h.c. = \hat{A}_x^{(+)}(z,t) + \hat{A}_x^{(-)}(z,t)$$

$$\hat{A}_x^{(+)}(z,t) = \sqrt{\frac{\hbar}{2V\varepsilon\omega_l}}e^{i\theta_l}e^{ik_l z - i\omega_l t}\hat{a}_l$$

$$\hat{A}_x^{(-)}(z,t) = \sqrt{\frac{\hbar}{2V\varepsilon\omega_l}}e^{-i\theta_l}e^{-ik_l z + i\omega_l t}\hat{a}_l^{\dagger}$$

$$\langle n_l|\hat{A}_x^{(-)}(z,t)\hat{A}_x^{(+)}(z,t)|n_l\rangle$$

$$\langle n_l|\hat{A}_x^{(-)}(z,t)\hat{A}_x^{(+)}(z,t)|n_l\rangle = \frac{\hbar}{2V\varepsilon\omega_l}\langle n_l|\hat{a}^{\dagger}\hat{a}|n_l\rangle = \frac{n_l\hbar}{2V_l\varepsilon\omega_l}$$

Wave of Arbitrary Polarization—Quantum Case

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{s\in\{v,h\}} \sqrt{\frac{\hbar}{2\omega\varepsilon V}}\mathbf{e}_s\hat{a}_s e^{i(kz-\omega t)} + h.c.$$

$$\hat{H} = \hat{H}_h + \hat{H}_v$$

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{s\in\{v,h\}} \sqrt{\frac{\hbar}{2\omega\varepsilon V}}\mathbf{e}_s\hat{a}_s e^{i(kz-\omega t)} + h.c. = \hat{\mathbf{A}}^{(+)}(\mathbf{r},t) + \hat{\mathbf{A}}^{(-)}(\mathbf{r},t)$$

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r},t) = \sum_{s\in\{v,h\}} \sqrt{\frac{\hbar}{2\omega\varepsilon V}}\mathbf{e}_s\hat{a}_s e^{i(kz-\omega t)}$$

$$\hat{\mathbf{A}}^{(-)}(\mathbf{r},t) = \sum_{s\in\{v,h\}} \sqrt{\frac{\hbar}{2\omega\varepsilon V}}\mathbf{e}_s\hat{a}_s^\dagger e^{-i(kz-\omega t)}$$

Quantum Surrealism:

$$|\Psi_{\text{one-photon}}\rangle = A_v|1_v\rangle|0_h\rangle + A_h|0_v\rangle|1_h\rangle = A_v|1_v\rangle + A_h|1_h\rangle$$

$$|A_v|^2 + |A_h|^2 = 1$$

$$M = \langle\Psi_{\text{one-photon}}|\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)|\Psi_{\text{one-photon}}\rangle$$

$$\langle 1_h|\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)|\Psi_{\text{one-photon}}\rangle = \frac{\hbar}{2V\varepsilon\omega}|A_h|^2$$

$$\langle 1_v|\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)|\Psi_{\text{one-photon}}\rangle = \frac{\hbar}{2V\varepsilon\omega}|A_v|^2$$

Polychromatic Photons vs Monochromatic Photons

$$\hat{A}_x(z,t) = \sum_l \sqrt{\frac{\hbar}{2V_l \varepsilon \omega_l}} \hat{a}_l e^{-i\omega_l t + ik_l z} + h.c.$$

$$\hat{H} = \sum_{l=1}^{\infty} \hat{H}_l$$

$$|\Psi_{\text{general}}\rangle = \prod_{l=1}^{\infty} |n_l\rangle \approx \prod_{l=1}^{N} |n_l\rangle$$

Single-Photon Case

$$|1\rangle_p = |0\rangle_1 \cdots |1\rangle_p \cdots |0\rangle_N$$

$$|\Psi^{(1)}\rangle \approx \sum_{p=1}^{N} \tilde{w}_p |1\rangle_p = \sum_{p'=1}^{N} \tilde{w}_{p'} \hat{a}^{\dagger}_{p'} |0\rangle = \hat{\mathbf{a}}^{\dagger} \cdot \tilde{\mathbf{w}} |0\rangle$$

$$\tilde{\mathbf{w}}^{\dagger} \cdot \tilde{\mathbf{w}} = 1$$

Two-Photon Case

- Unentangled Two-Photon Case

$$|\Psi^{(2)}\rangle = |\Psi_A^{(1)}\rangle \otimes |\Psi_B^{(1)}\rangle = \left(\hat{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{w}}_B\right)|0\rangle \otimes \left(\hat{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{w}}_A\right)|0\rangle = \left(\hat{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{w}}_B\right)\left(\hat{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{w}}_A\right)|0\rangle$$

$$\langle\Phi_A|\Psi^{(2)}\rangle$$

$$\langle\Phi_A|\Psi^{(2)}\rangle = \left(\langle\Phi_A|\Psi_A^{(1)}\rangle\right) \otimes |\Psi_B^{(1)}\rangle$$

Two-Photon Case

- Entangled two-photon case

$$|\Psi^{(2)}\rangle = \sum_i \sum_j \Psi_{i,j} \hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle$$

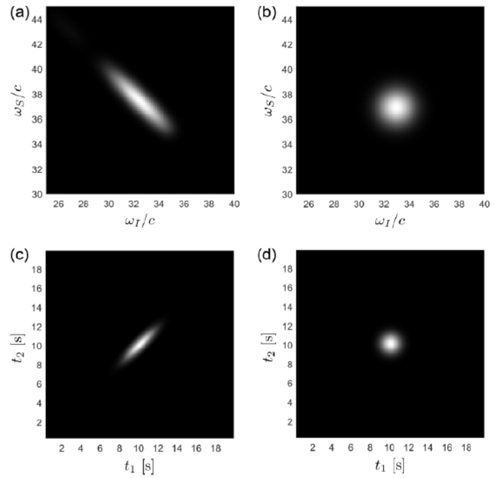


Figure 39.4: Spectral probability amplitudes for (a) two entangled photons and (b) two nonentangled photons. The degree of two-time coincidence count for (c) entangled and (d) nonentangled photon pairs. The entangled photon pairs both strong temporal correlation and frequency anti-correlation, but not for the non-entangled photon pairs [58].

Conclusions

- **Overview of quantum electromagnetics and their derivations are given;**
- **Quantum pendulum is used to understand quantum EM;**
- **Hamiltonian theory based on energy conservation is invoked;**
- **Quantum Maxwell's equations look strikingly similar to classical Maxwell's equations;**
- **QME allow the assignment of quantum states and study the discreteness of photons;**
- **They allow the study of quantum interference, quantum entanglement, and other quantum weirdness;**
- **Advancement in quantum technologies will need more CEM in the future, or solutions to quantum Maxwell's equations;**
- **A larger knowledge base is needed to advance this field.**

PURDUE
UNIVERSITY

Mung Chiang
Purdue President
48 yr old, from
HK SAR

Purdue Mall

Purdue U Gate

Neil
Armstrong
Purdue
graduates
the largest
number of
astronauts

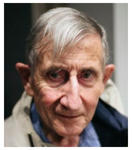
Strong
emphasis
On
Diversity,
Equity,
Inclusion.

Final Words of Wisdom

The best way to win a war is to have no war
---Sun Zi, 500BC

Use ICT to bring about a more peaceful world.

- Advancing Technology for Humanity—Leah Jamieson (1949-) (Former Purdue Dean of Engineering, IEEE President).
- "Technology is a gift of God. After the gift of life, it is perhaps the greatest of God's gifts. It is the mother of civilizations, of arts and of sciences."—Freeman Dyson (1923-2020).



- All Humans Can Be Taught—Confucius (450BC).

- Love your enemy—Buddha (500BC) and Jesus Christ (20 AD).

- Our human mind is the greatest gift of God!
- Peace to this world. Collaborate to solve our global warming problems and save our planet Earth!